Odd and Even Functions Revisited: A Symmetry View of the Walsh-Hadamard Transform

Most students first encounter **odd and even functions** in elementary calculus:

- Even: $f(-x) = f(x)$
- Odd: $f(-x) = -f(x)$

After learning how to decompose a function into odd and even parts, the topic often seems to disappear from mathematics. In reality, the underlying idea becomes far more important—it simply reappears under the language of **symmetry**, **linear algebra**, and **group representations**.

This note explores an elegant connection between parity decomposition and the Walsh-Hadamard transform.

The familiar one-dimensional decomposition

Every function can be written uniquely as

$$f(x) = f_e(x) + f_o(x),$$

where

$$f_e(x) = \frac{f(x) + f(-x)}{2},$$

and

$$f_o(x) = \frac{f(x) - f(-x)}{2}.$$

Rather than viewing these as algebraic tricks, it is useful to think of them as **projections onto symmetry classes**.

Reflection is an operator

$$R : f(x) \to f(-x).$$

Since reflecting twice returns the original function,

$$R^2 = I,$$

its only possible eigenvalues are

- +1 (even functions)
- -1 (odd functions)

Thus odd and even functions are simply the two eigenspaces of the reflection operator.

Two dimensions introduce four symmetry classes

For a function $f(x, y)$, reflections may occur independently in each coordinate.

There are now two commuting reflection operators:

- $R_x : (x, y) \to (-x, y)$
- $R_y : (x, y) \to (x, -y)$

Each reflection contributes an independent parity.

The result is four possible symmetry classes:

x parity	y parity
Even	Even
Even	Odd
Odd	Even
Odd	Odd

Every sufficiently well-behaved function decomposes uniquely into these four components.

This is no longer merely "odd and even functions." It is already the decomposition of a vector space into simultaneous eigenspaces of multiple commuting symmetry operators.

Looking at four reflected samples

Now sample the function at the four reflected locations

$$a = f(x, y)\ b \quad = f(-x, y)\ c = f(x, -y)\ d \quad = f(-x, -y).$$

These four values completely capture the local parity behaviour around the point.

The natural question becomes:

Can we directly extract the amount of each parity present?

The answer is yes.

Enter the Walsh-Hadamard transform

The four-point Walsh-Hadamard transform computes

$$A = a + b + c + d\ B \quad = a - b + c - d\ C = a + b - c - d\ D \quad = a - b - c + d.$$

Each output corresponds to one symmetry class.

Coefficient	Interpretation
A	Even in x, Even in y
B	Odd in x, Even in y
C	Even in x, Odd in y
D	Odd in x, Odd in y

The familiar Hadamard matrix therefore performs much more than a collection of additions and subtractions.

It converts samples indexed by reflected points into coefficients indexed by parity.

The Hadamard basis

The four Walsh basis vectors are

$$(1, 1, 1, 1)$$
$$(1, -1, 1, -1)$$
$$(1, 1, -1, -1)$$
$$(1, -1, -1, 1).$$

Each vector represents a distinct pattern of sign changes under reflection.

Rather than arbitrary ±1 patterns, they are precisely the common eigenvectors of the two reflection operators.

This provides a natural geometric interpretation of the Hadamard basis.

Generalization

Nothing special happens in two dimensions.

Each additional coordinate contributes another independent reflection.

Therefore:

- 1D → 2 parity classes
- 2D → 4 parity classes
- 3D → 8 parity classes
- nD → 2^n parity classes

The corresponding Walsh-Hadamard transform has exactly 2^n basis vectors.

Each basis vector corresponds to one combination of parity under the independent coordinate reflections.

A group-theoretic viewpoint

The reflections form the binary symmetry group

$$(\mathbb{Z}_2)^n,$$

where each coordinate contributes one binary reflection.

The Walsh basis functions are the **characters** (one-dimensional irreducible representations) of this group.

Consequently, the Walsh-Hadamard transform diagonalizes all coordinate reflections simultaneously.

This is directly analogous to how the Fourier transform diagonalizes translation operators.

Viewed this way:

- Fourier analysis studies translation symmetry.
- Walsh-Hadamard analysis studies reflection symmetry.

Why this viewpoint is useful

Thinking in terms of symmetry rather than arithmetic helps explain why the Walsh-Hadamard transform appears in so many areas:

- fast transforms
- signal processing
- coding theory
- random projections
- locality-sensitive hashing
- Boolean function analysis
- quantum computing
- machine learning

The transform is not merely fast—it is adapted to the binary symmetry structure of the problem.

Final thought

Odd and even functions are often presented as an isolated topic in introductory mathematics.

A more revealing perspective is that they are the simplest example of decomposing a space according to the eigenvalues of symmetry operators.

In one dimension this produces two parity classes.

In multiple dimensions it produces 2^n symmetry classes.

The Walsh-Hadamard transform is precisely the change of basis that exposes those classes, converting local function values into independent parity components.

Seen from this perspective, the Walsh-Hadamard transform is not simply an efficient algorithm—it is the natural spectral decomposition of reflection symmetry.